

Quantum randomness and free will

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Both deterministic and indeterministic physical laws are incompatible with control by genuine (non-illusory) free will. We propose that an indeterministic dynamics can be *weakly* compatible with free will (FW), whereby the latter acts by altering the probability distribution over allowed outcomes. In the quantum physical world, such a FW can collapse the wave function, introducing deviations from the Born rule. In principle, this deviation would stand in conflict with both special relativity and (a variant of) the Strong Church-Turing thesis, implying that the brain may be an arena of exotic, non-standard physics. However, in practice, these deviations would not be directly or easily observable, because they occur in sub-neuronal superpositions in the brain, where they would be shrouded in random measurement errors, noise and statistical fluctuations. Our result elucidates the difference between the FW of human observers and that of observed particles in the Free Will Theorem. This difference is a basic reason for why FW (and, in general, consciousness) cannot be recreated by standard artificial intelligence (AI) technology. We propose various neurobiological experiments to test our proposed theory. We speculate that for observers to be aware of a physical theory such as quantum mechanics, FW is necessary and that the theory must therefore not be universal. We suggest that FW may be regarded as a primitive principle in Nature for explaining quantum indeterminism.

I. INTRODUCTION

Free will (FW), as we normally understand it in daily life, is the power to make one's own choices. Our outlook on the world implicitly assumes that human behavior is governed by FW: we choose, we plan and we normally hold people responsible for what they say or do. Recently, a number of physicists have studied FW in the context of the foundations of quantum mechanics (QM), in particular, in relation to non-classical features like indeterminism, entanglement, non-realism and contextuality [1–9].

Determinism means that the past history of the universe completely determines the future of a system, leaving no room for FW to manoeuvre it. Hence, unless FW is assumed to be illusory, deterministic laws of physics are logically incompatible with the action of free-willed agents. Neither is randomness good for FW. Evolution is random if the physical laws are indeterministic, so that the past history of the universe does not determine completely the future evolution of a system. Nevertheless, this “freedom” will be governed by some probability rule, like the Born rule of quantum mechanics (QM). If there is no rule to choose from among possible alternatives, one would still expect a default, democratic rule of all alternatives being equally probable or an appropriate noise pattern. The frequencies of outcomes would be constrained by the law of large numbers applied to the relevant rule. This would exclude the possibility of full control, and hence of FW as we normally understand it. Therefore, the “free” (or “random”) aspect of FW stands in conflict to the “will” (or “control”) aspect, making the word “free will” an oxymoron. We call this the weak FW paradox (WFWP).

Informally, WFWP says that if we accept FW axiomatically, then FW can coexist peacefully neither with determinism nor randomness in physical reality. A strong version of the FW paradox (SFWP) asks whether, in this light, such a thing as FW exists at all or is definable. Suppose we wish to argue that a quantity x (say, the position of one's hand) is free-willed. To make the case that x is not random, but determined entirely by one's voluntary decision, we write $x = x(\mathbf{w})$, where \mathbf{w} is the ‘intention variable’. To ensure that x does not thereby become deterministic, we require that \mathbf{w} is free and not determined by the past history of the universe. But this would \mathbf{w} , and by consequence x , as random. So we have to assume that \mathbf{w} itself, like x , is determined by other ‘lower level’ variables. An unending regression of this sort is obvious. From this standpoint, determinism and pure randomness seem to be the only fundamental primitives to construct a dynamics; FW as yet another primitive does not seem to exist, or at rate, seems to be undefinable in the conventional language of physics. SFWP alerts us to the fact that if FW exists, then it must be sought in the middle-ground between randomness and determinism, and this, in a way to be made rigorous below, will be our approach.

The remaining article is arranged as follows. In Section II, we present a possible resolution of WFWP, based on

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a clue from Tarski’s semantic theory of truth, and a partial solution to SFWP, a more detailed treatment of which will be taken up elsewhere [10]. The implication of our work for the Free Will Theorem [1] is discussed in Section III. The question of compatibility with relativity and efficient computability is studied in Section IV. In Section V, we consider the neuroscientific implications and experimental tests of our proposal. Philosophical and metamathematical ramifications are taken up in Section VI, before concluding in the final section. We will refer to WFWP and SFWP together as “FWP”. Throughout the article, we will find it convenient to adopt a Copenhagenesque interpretation of QM and talk of “wavefunction collapse”, “state vector reduction”, “the quantum measurement problem” and the quantum-classical divide, mainly because this terminology is convenient for our purpose and is used informally by working physicists. Any imprecision in the usage does not affect our arguments. We indicate our preferred position in this regard later.

II. THE FREE WILL PARADOX AND CONSEQUENCES

FWP does not arise if it assumed that FW is illusory and that choices are made deterministically in the sub-conscious and uploaded to the conscious mind that is unaware that it lacks true freedom to choose [5, 6]. For our purposes, by “FW”, we will always mean genuine— and not illusory— FW, unless explicitly otherwise qualified. Another resolution to WFWP [7] could be that, though conforming to the required probability distribution, FW has room to alter the *ordering* of outcomes. This would require that free-willed outcomes (say a stream of bits) would have to be compensated by uncontrolled outcomes at other times to restore eventual conformance with the probability distribution.

Our proposed resolution of FWP is based on an insight due to Tarski for resolving the liar paradox [11], closely related to Gödel’s incompleteness theorem [12]. The liar paradox, which self-referentially states *this statement is false* is inconsistent because it is true if and only if it is false. To avert the paradox, it is necessary to distinguish between the language of discussion (called the *object language* L^0) from the *meta-language* (L^1), in which to talk *about* L^0 . The semantic truth of statements in L^0 cannot consistently be asserted in L^0 itself, but in L^1 . It is the failure to make this distinction between L^0 and L^1 that leads to the liar’s paradox.

Analogously, in the present situation, we posit a basic or zeroth level of physical reality, called *objective reality* and denoted \mathbf{R}^0 , the arena where the standard laws of quantum mechanics govern physical or *objective* quantities and variables like the position and momentum of particles. This is distinguished from a meta-level or first level of reality, called *subjective reality* and denoted \mathbf{R}^1 , the arena where some other laws of Nature govern certain *subjective* quantities and variables, like FW, emotions and thoughts of sentient beings like human agents.

Any procedure outputting an *objective* result (one describable in the language of standard quantum physics) is called an objective-valued (OV) procedure. An experimenter selecting a detector setting, or a computer printing out a read-out are examples of OV procedures. We can consistently characterize FW at the objective level as follows. An OV action is said to be free-willed if (a) it is not completely determined by the past *objective* history of the universe; and (b) it depends explicitly on some *subjective* variables.

Condition (a) implies that the outcome of the action is random at the objective level, or \mathbf{R}^0 -random. Condition (b) leaves room for the subjective influence/control of the action. FW can thus be said to simultaneously and consistently have both freedom and control by recognizing that the freedom is with respect to \mathbf{R}^0 , whereas control is with respect to \mathbf{R}^1 . The distinction between the subjective and objective realities is, in this way, the key to solving WFWP, and highlights the fact that it is not possible to define FW acting on \mathbf{R}^0 *within* standard QM, the language/dynamics of \mathbf{R}^0 .

For our present purpose, a physical particle is taken to be a purely objective entity, living in \mathbf{R}^0 ; by contrast, a human being, such as an observer measuring a photon’s polarization, is a subjective-objective entity. In particular, she/he is characterized by subjective degrees of freedom (specifying thoughts, feelings and FW) that dynamically couple in some so-far unknown way to certain objective degrees of freedom in the brain. In consonance with the terminology of Cartesian dualism, we call the subjective component as her/his *mind*, and the objective component her/his *body* [13]. What we have called ‘objective’ and ‘subjective’, are, traditionally in classical metaphysics, referred to as ‘physical’ and ‘non-physical’, respectively.

In the sense specified, a particle has no mind. The ‘FW’ of a particle can only satisfy condition (a), but not (b). It has freedom but no control. The particle’s FW is simply pure randomness. FWP thus does not arise in the case of particles. To normalize the terminology, we will call this particulate FW as *zeroth-order* FW (zeroth-order freedom and no control), in contrast to the FW of human agents, which satisfies both conditions (a) and (b) and will henceforth be called *first-order* FW (first-order control over zeroth-order freedom). An entity with first-order FW necessarily also has zeroth-order FW, as it can choose to run or not run a quantum random process, (e.g., measuring σ_x on a qubit prepared in an eigenstate of σ_z) as a subroutine. (In fact, human agents possess second-order FW, a point we return to elsewhere [10].) To avoid repetitiveness, in the remaining article, we sometimes drop the qualifications ‘first-order’

or ‘zeroth-order’ when the context makes it clear which of these two is meant.

It remains to understand the mechanism of how first-order FW influences objective degrees of freedom in physical reality. This will automatically help solve the mind-body problem of dualism, which is the absence of an empirically identifiable *mind-body interface*, or a meeting point between the mind and body. The key observation here is that probabilistic laws governing \mathbf{R}^0 will in general be compatible with FW only if the control feature (b) is absent. This implies that first-order FW, which incorporates (b), will couple to objective degrees of freedom *by causing deviations from the otherwise-expected objective probability distributions*.

An illustration: suppose X is an OV random variable with some probability distribution P , and is influenced by the free-willed intervention of some agent. Given any sequence of n tosses of X , represented by $\underline{X} \equiv X_1 X_2, \dots, X_n$, such that the probability of occurrence $P(\underline{X}) > 0$, then FW can force the occurrence of outcome \underline{X} without *logical* contradiction. That is, it can realize any possible evolutionary branch allowed by the indeterministic evolution rules of the system, but cannot realize any not allowed by the rules. By controlling the evolution to be within the space of allowed branches, FW does not stand in direct conflict with the rule specified by P , in a sense, even though in the long run, such FW-induced interventions will become manifest as departures from P . We call this relation between FW and P , whereby free-willed interventions contravene P statistically, but not logically, as the *weak compatibility* of FW and P .

In specific, if X is the random variable corresponding to a FW-influenced quantum measurement \mathcal{M} on system S , then X may deviate from the Born probability rule (that this rule represents genuine freedom of particles, under certain reasonable assumptions, is shown in Section III). For example, if S is a qutrit (three-level quantum system) in the state $|\psi\rangle \equiv \cos\theta|0\rangle + \sin\theta|1\rangle + 0|2\rangle$, then FW can control and direct the collapse of $|\psi\rangle$ under \mathcal{M} to yield $|0\rangle$ or $|1\rangle$ with certainty, or at any rate, with probabilities other than the standard Born probabilities of $\cos^2\theta$ and $\sin^2\theta$, respectively. However, outcome $|2\rangle$ is forbidden by virtue of weak compatibility. FW will therefore manifest as *statistical deviations from the Born rule* over allowed outcomes. By contrast, zeroth-order FW, which is plain quantum randomness, will conform to the Born rule in the long run.

From an objective perspective, the wavefunction is *intangible*. It cannot be manipulated or controlled in ways that depend on its value. For example, an amplitude cannot be abruptly set to zero. Mathematically, this makes QM a linear theory. By contrast, the FW-directed collapse is a nonlinear phenomenon. This makes the wavefunction a subjective *tangible*, as manifested by the control FW has over it. To wit, one may think of the wavefunction as a ghost that is physically insubstantial, but quite substantial in the subtler sub-physical world. The wavefunction can then be fancifully described as an informational ghost that bridges the subjective and objective worlds across the mind-body interface.

Presumably, FW is exercised as an observer’s subjective control over the collapse of certain sub-neuronal superposition states in the brain under special conditions available there. It is assumed that these conditions also determine the specific basis of collapse. Subjective degrees of freedom (characterizing the agent’s volition, thoughts, etc.) couple dynamically in some so-far unknown way to objective degrees of freedom (pertaining to motor neurons deterministically associated with various voluntary bodily functions) through this mind-body interface. We note that though the exercise of FW is conscious, the mechanism by which it controls state vector reduction, and its subsequent deterministic amplification to macroscopic classical actions, will itself be at a sub-conscious level.

We will find it convenient, following the practice in Vedantic philosophy, to identify the mind as the subjective counterpart of the brain areas corresponding to emotions, memory and FW, and identify the *intellect* as the subjective component of the ratiocinative and determinative faculty in the brain [14]. The model of free-willed action that we propose is the following three-staged process:

Stage 1— Attention. Faced with a situation that requires a choice to be made from among various alternatives j , the brain momentarily creates a sub-neuronal quantum superposition $|\psi_\alpha\rangle = \sum_j \alpha_j |j\rangle$, which is presented to the mind. (Based on the idea presented in Ref. [15], brain microtubules may be the seat of such superpositions.) The coefficients α_j of $|\psi_\alpha\rangle$ may reflect certain priorities set by the physical brain.

Stage 2— Selection. The mind transfers the quantum information $|\psi_\alpha\rangle$ to the intellect (Why this intermediate step is required is explained elsewhere [10].) The intellect evaluates the value of the alternatives j according to some norm (emotional, social, etc.), choosing the optimal one, J , from among them. If the norm does not enable the choice of one of the alternatives, a random choice could be made (by invoking a neural zeroth-order FW subroutine). That the selection may be influenced, but is not determined, by the amplitudes of the superposition, is at the origin of departures from the Born rule. The intellect conveys the choice J to the mind.

Stage 3— Collapse. The mind exercises FW to direct the quantum state $|\psi_\alpha\rangle$ to collapse the quantum superposition to the state $|J\rangle$ corresponding to the chosen alternative.

We refer to this model as the “ASC” (the initials of “Attention-Selection-Collapse”) model. Because the selection in stage 2 is not according to the Born rule, stage 3 will in general cause a violation of energy conservation: a

measurement described by projective operators M_j non-selectively transforms the density operator ρ according to $\rho \rightarrow \rho' \equiv \sum_j M_j \rho M_j^\dagger$, assuming the Born or trace rule for probability. This transformation preserves energy, i.e., $\text{Tr}(\mathcal{H}) = \text{Tr}(\mathcal{H}\rho')$ if the measurement operator $\mathcal{M} \equiv \sum_j m_j M_j^\dagger M_j$ commutes with the system Hamiltonian \mathcal{H} . This is the case for quantum non-demolition (QND) measurements [16] but in general not true, when the interaction is dissipative and energy is exchanged with the environment [16].

Even if we assume, conservatively, that the basis in which FW-induced collapse happens in the energy basis and thus commutes with \mathcal{H} , the Born deviation implies that in general, energy will not be conserved on average. This is not a cause for worry because the violation of energy conservation would occur in a small sub-cellular region of the motor cortex of the brain, where it will be hardly discernible from measurement errors and the surrounding nervous noise. Moreover, the remaining features of the brain's physiology can be described in terms of deterministic observable causal chains obeying the conventional physical conservation laws (e.g., the metabolism involved in the arousal potentials triggering bodily movements) [7]. This fact also makes observing FW-induced effects directly difficult (cf. Section V).

The immediate main gain from the ASC model is that it allows us to formally distinguish inanimate entities like particles from animate entities like human observers, from the viewpoint of their degree of freedom and conscious initiative or control. To illustrate this we will consider two classes of entities, which are externally or objectively similar, but are distinguishable formally according to the above model.

Definition 1 *A Constantly Good Person (CGP) is an entity that, when presented with a good and bad alternative of action, evaluates their moral value, and exercises his FW to select the good alternative.*

We define a binary moral-valued function $\mathcal{M}(\mathbf{x})$ by $f : \Xi \mapsto \{\text{“good”}, \text{“bad”}\}$, where Ξ is the space (of description in some language) of actions in real-life contexts. It is straightforward to generalize this to a real-valued moral-valued function where, in some fashion, the moral value is quantified.

Definition 2 *A Noble Robot (NR) is a (quantum) algorithm that, when presented with a good and bad alternative of action, computes their moral value $\mathcal{M}(\mathbf{x})$, and selects the good alternative.*

In objective terms and the language of standard QM, a CGP is difficult to distinguish from an NR. As the mind is an unobservable mental agency affecting the random dynamics of objective variables, and the quantum wave function is intangible, the ontological status of both these items remains unclear in standard QM. At best we may find certain neural correlates in the CGP's brain suggestive of genuine moral judgment, but we can never conclusively argue that he is not an NR of some sort. However, it is clear that the richer language required in the ASC model entails that the two systems have distinct descriptions at a subjective level. This forms the content of the following theorem.

Theorem 1 *A CGP and NR are first-order distinguishable.*

Proof sketch. When a NR's logical processor computes the moral value of the presented two alternatives, only the “good”-valued alternative triggers a switch that controls the NR's motor circuit. By contrast, presented with the alternatives, a CGP's brain creates a quantum superposition of the alternatives (stage 1). CGP's intellect determines the good alternative, and the mind directs the wavefunction collapse to the state corresponding to this alternative. The NR is algorithmically and dynamically bound to choose the good alternative. The idea that CGP is not bound by the objective laws of physics and freely chooses the good alternative is represented by the superposition state, which implies that the CGP *could* have chosen otherwise but *would* not. ■

Theorem 1 can be extended to indicate why AI technology, even with access to standard quantum randomness, cannot recreate FW. Although an AI algorithm for an NR can, to a good extent, simulate a CGP, it is qualitatively different. More generally, one can define an *inconsistently noble robot*, where a probabilistic selection procedure involves the creation of a superposition and measurement in the computational basis. By contrast, an *inconstantly good person* would require quantum dynamics with deliberate, subjective intervention. These considerations entail that a human agent, and by extension any sentient agent, could not be considered merely as a sufficiently complex robot, but a qualitatively distinct class of entities. Here Penrose's interesting thesis is worth noting, according to which conscious processes are fundamentally non-algorithmic [17], an idea that is implicit in ancient East Asian philosophies.

III. IMPLICATIONS FOR THE FREE WILL THEOREM

Zeilinger notes that there are two “freedoms”: that on the part of Nature, and that on the part of the experimental observer [18]. Echoing this idea more rigorously, the FW theorem (FWT) [1], which is essentially a non-locality proof for any theory that purports to reproduce quantum correlations, states that if observers have FW, then so too do

S_1	(0, 0, 0, 1)	(0, 0, 1, 0)	(1, 1, 0, 0)	(1, -1, 0, 0)
S_2	(0, 0, 0, 1)	(0, 1, 0, 0)	(1, 0, 1, 0)	(1, 0, -1, 0)
S_3	(1, -1, 1, -1)	(1, -1, 1, 1)	(1, 1, 0, 0)	(0, 0, 1, 1)
S_4	(1, -1, 1, -1)	(1, 1, 1, 1)	(1, 0, -1, 0)	(0, 1, 0, -1)
S_5	(0, 0, 1, 0)	(0, 1, 0, 0)	(1, 0, 0, 1)	(1, 0, 0, -1)
S_6	(1, -1, -1, 1)	(1, 1, 1, 1)	(1, 0, 0, -1)	(0, 1, -1, 0)
S_7	(1, 1, -1, 1)	(1, 1, 1, -1)	(1, -1, 0, 0)	(0, 0, 1, 1)
S_8	(1, 1, -1, 1)	(-1, 1, 1, 1)	(1, 0, 1, 0)	(0, 1, 0, -1)
S_9	(1, 1, 1, -1)	(-1, 1, 1, 1)	(1, 0, 0, 1)	(0, 1, -1, 0)

TABLE I: List of 9 observables Alice can measure, out of 18 possible directions.

observed particles. A brief account of FWT, based on a test of quantum nonlocality given in Ref. [19], is as follows. Consider 9 orthogonal bases in a Hilbert space of dimension 4, and denote them by S_1, S_2, \dots, S_9 . Each S_j is a set of 4 orthogonal vectors, denoting directions of detector settings, which we write in un-normalized, ‘row’ representation as $(1, 0, 0, 0), (1, -1, 0, 0) \equiv \frac{1}{\sqrt{2}}((1, 0, 0, 0) - (0, 1, 0, 0))$, etc.

Measuring S_j on a state that is not its eigenstate, will collapse it to an outcome randomly according to the Born rule. This randomness by itself does not at first imply the freedom of the particle. Measurement only *finds* the outcome random. Unknown causes from the past may well have made it determinate prior to measurement, the randomness being an artefact of our not knowing those causes. The Kochen-Specker theorem [20] rules out that any such determinate assignment of outcomes exists independent of context (here the S_j ’s).

A proof of the Kochen-Specker theorem is as follows. Suppose a determinate assignment exists for each vector appearing in Table I. Thus in each row of Table I, precisely one of the four vectors is to be assigned the value 1, and the remaining three the value 0 (corresponding to the SPIN assumption of FWT). The same value (0 or 1) is to be assigned to all instances of a vector: eg., (1000) is assigned the same value in S_1 and S_2 . But a careful look at the Table shows that no such assignment scheme exists. For the sum of assignments across each row is 1, yielding 9 as the sum of values for each row. But then each vector occurs precisely twice over all rows taken together. Thus the sum of assignments of all vectors should be an even number. This contradiction implies that no determinate assignments can be made to the vectors, except if they are context-dependent. This means that two instances of the same vector, e.g., $(1, 0, 0, 0)$ occurring in S_1 and S_2 , need not have the same value. This impossibility to pre-assign values means either that particles have freedom, or their values could be predetermined in a context-dependent way.

Therefore, to show that particles have freedom, we need to create a situation where information about context will not be available during measurement. To this end consider Alice and Bob sharing entanglement of the form:

$$|\Psi_J\rangle = \frac{1}{2} \sum_k |S_J^{(k)}\rangle_A |S_J^{(k)}\rangle_B, \quad (J = 1, \dots, 2). \quad (1)$$

It can be shown that this form is preserved in any of the above bases, i.e., $|\Psi_1\rangle = |\Psi_2\rangle = \dots = |\Psi_9\rangle$. This ensures that if Alice and Bob measure in the same basis S_j , they will find the same outcome k (corresponding to the TWIN assumption of FWT).

Alice randomly measures one of the 9 bases, S_j , which will serve as context. Denote her outcome as $|k'\rangle$, to which she assigns value 1, and 0 to the other elements of S_j . Bob independently and randomly chooses one of the 18 vectors k appearing in Table I and measures the binary observable $|k\rangle\langle k| - (\mathbb{I} - |k\rangle\langle k|)$. He assigns $|k\rangle$ the value 1 or 0 depending on whether he detects it or not. QM guarantees that if Bob’s vector k happens to lie in the context S_j they will assign it the same value 0 (in case of non-detection of $|k\rangle$) or 1 (in case of detection). If we try to explain the above in terms of prior assignment of values, Alice should be able to communicate her context information j to Bob. By ensuring that Alice’s and Bob’s measurements are sufficiently close in time, we can rule out such context communication at any finite speed (FWT’s FIN assumption, later weakened to MIN). If relativistic invariance is assumed (though this is not necessary), then their measurement events are required to be spacelike separated.

We have thus far implicitly assumed that the observers have freely and randomly selected their detector settings. It is still possible that there is no freedom of the particles, provided that the experimental settings and the outcomes are pre-determined in such a way as to suggest stronger-than-classical correlation, and that the FW of the experimenters is an illusion. (An observation that weakens that possibility is the question of why such a conspiratorial hyper-determinism should go only so far as to suggest stronger-than-classical correlations but stop short of suggesting

superluminal communication.) Therefore, if we assume that the experimenters have genuine FW, then all avenues of attributing determinate values to Alice’s and Bob’s outcomes (no matter what the unknown causes) are ruled out, and they must be considered as a genuine acts of creation or FW on the part of the particles/Nature.

The above proof of FWT is not suggestive of any control by experimenters on the randomness they generate. Their FW is not required to be any different from the zeroth-order FW of observed particles. However, by virtue of Theorem 1, we expect the experimenters’ FW to be of first-order. We can envisage that (for example) the experimenters are graduate students whose work is affected by adverse comments on their earlier run of trials, or plans for a exciting week-end mountaineering trek. This could manifest as fluctuations in the random settings they chose, which may be detectable by certain standard randomness tests. This suggests some kind of control and hence that the FW exercised by the experimenters is of first-order: the ‘particle’ in the observer’s brain possibly creates equal amplitude superpositions for the alternatives. But the emotional vulnerability of the mind affects the choice of the state selected for collapse.

The control aspect of the FW of human agents could manifest as large-scale deviation from the expectation value of some quantity. For example, if the expected behavior based on applying the Born rule in stage 2 of the ASC model dictates following the local gradient in a brain energy-potential landscape (e.g., to eat food placed in front of a person), the person the brain belongs to may exercise FW to deviate from this behavior (e.g., to fast in the interest of good health in the long run or in order to feed an indigent person). Thus, if the control aspect of FW were ‘switched off’, degrading the FW to zeroth-order, then a purely random, zombie-like behavior would result, where agents’ behavior will lack any self-control, long-term planning as well as friendly/altruistic behavior. FWT as it stands neither requires nor implies this difference between the FWs of observers and particles. Therefore, it does not preclude that human FW can be recreated with AI technology having access to standard quantum randomness. Our distinction between the two orders of FW preclude this possibility. Perhaps FWT may be recast as a moral dilemma in order to highlight the difference. We will revisit this issue in a subsequent work [10].

IV. IMPLICATIONS FOR SPECIAL RELATIVITY AND COMPUTATIONAL COMPLEXITY

The laws of physics can be characterized by no-go conditions, which specify operations that are physically impossible. Two such basic conditions that have been recognized in quantum information science are the no-signaling condition and the so-called “the world is not hard enough” (WNHE) assumption, an empirical rule-of-thumb based on a computer scientific survey of the computational power of physical systems [21, 22]. We expect them to be true because of, respectively, special relativistic restriction and our notion of what are reasonable models of computation. As shown in the following two subsections, both these no-go conditions are violated by non-Bornian probabilities, essentially because the FW-induced effects make QM nonlinear. However, as with energy non-conservation, they would happen at the sub-cellular level within the brain. As a result, they would be difficult to observe directly, and are certainly not contradicted by known empirical verifications of these conditions. In principle, however, these observations imply that the brain is a seat of exotic and highly non-standard physics, presenting a potential for uncovering new physics.

A. Special relativity considerations

In a multipartite quantum system, any quantum operation applied locally to one part does not affect the reduced density operator of the remaining part. This fundamental no-go result, called the “no-signalling theorem”, implies that quantum entanglement does not enable nonlocal (“superluminal”) signaling under standard operations, and is thus consistent with special relativity. This peaceful coexistence of quantum nonlocality and relativity is arguably tense [9] because of the violation of Bell-type inequalities [23], but it does not overtly imply spacelike influence. Usable information always requires the transport of a material object and hence cannot be communicated outside the lightcone, in agreement with relativity theory.

The ambiguity between whether this spooky spacelike signaling is for real or not, would be broken, and superluminal communication would become a reality, if measurement outcomes can be forced on superpositions, in violation of the Born rule [22]. For example, if agents Alice and Bob share the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and Alice is able to harness FW to deterministically collapse her qubit to $|0\rangle$ or $|1\rangle$, then she can control whether to leave Bob’s qubit in the state $|0\rangle$ or $|1\rangle$, respectively, and thus signal Bob 1 bit superluminally. This fact remains even if she has only partial control over the collapse, so long as there is a deviation from the Born rule (see Ref. [22] and references therein). The signaling speed attainable in this manner is arguably the ‘speed of quantum information’, namely the lower bound on the speed at which a (superluminal) signal should travel in order to explain the violation of Bell-type inequalities [23]. In the currently available experimental estimate, this speed is at least of the order of $10^4 c$ [24].

Therefore, if we accept the reality of FW, and FW-induced collapse, relativity must be considered as an approximate description pertinent only to *objective reality*, and the apparent absence of superluminal signals is attributed to the lack of macro-scale quantum superpositions with which subjective variables can interface. From the perspective of subjective (‘higher’) reality, space, time and causality are taken to be absolute. Subjective quantities like first-order FW and states of mind (emotions and intentions) are frame-independent.

Consider a Bell-type experiment of a pair of entangled particles, where two observers Alice and Bob freely and independently measure their respective particle at spacelike-separated events. Further let Alice and Bob be in relative motion such that each considers her/his own measurement as the first [25]. An explanation of the violation of Bell-type inequalities in terms of a superluminal signal traveling from one observer’s measurement event to the other’s is untenable in the relativistic causal description, because of the incomparable time-ordering of the events. In our formalism, since the subjective, absolute perspective is taken to be the right picture, and the objective, relativistic perspective the approximate picture, invoking superluminal influence to explain quantum correlations poses no logical contradiction. One of Alice and Bob will be deemed as the first to measure and the originator of the superluminal signal, though this fact makes no difference to an observer who interprets their quantum correlations in the relativistic framework.

One might consider an experimental test of our model of looking for a inertial reference frame, potentially the absolute frame, such that simultaneous measurement in that frame leads to a break-down in the quantum correlations of the TWIN kind. However, such a break-down need not happen, if instantaneous communication over finite distance were possible. Such infinitely fast communication would be less surprising if we regard space, not as a physical barrier, but as a kind of information, and thus ultimately not unlike an internal degree of freedom [22].

The very small, sub-neuronal scale at which such FW-induced violation of no-signaling occurs ensures that it is not invalidated by available neurobiological data and experimental tests in support of Relativity. At the same time, this will present challenges for its experimental observation, as discussed in Section V. This possible FW-induced violation of no-signaling is not in conflict with the MIN assumption of FWT, provided the ‘speed of quantum information’ is finite. But even otherwise, this mechanism of superluminal signaling presumes FW on the part of the observer and particle, the very features FWT either assumes or purports to prove.

B. Computational complexity considerations

The central problem in theoretical computer science is the conjecture that the computational complexity classes, **P** and **NP**, are distinct in the standard Turing model of computation. **P** is the class of decision problems efficiently (that is, in time that grows at most polynomially as a function of problem input size) solvable by a (deterministic) Turing machine (TM). **NP** is the class of decision problems whose solution(s) can be verified efficiently by a deterministic TM. The word “complete” following a class denotes a problem X within the class, which is maximally hard in the sense that any other problem in the class can be solved in polynomial time if there is an algorithm that solves X efficiently. For example, determining whether a Boolean formula is satisfied is **NP**-complete.

P is often taken to be the class of computational problems which are “efficiently solvable” (i.e., solvable in polynomial time) or “tractable”, although there are potentially larger classes that are considered tractable such as **BQP**, the class of decision problems efficiently solvable by a quantum computer. **NP**-complete and potentially harder problems, which are not known to be efficiently solvable, are considered intractable in the TM model. If $\mathbf{P} \neq \mathbf{NP}$ and the universe is a polynomial- rather than an exponential- place, physical laws cannot be harnessed to efficiently solve intractable problems, and **NP**-complete problems will be intractable in the physical world.

That classical physics supports various implementations of the Turing machine is well known. More generally, we expect that computational models supported by a physical theory will be limited by that theory. Empirical evidence suggests that the physical universe is indeed a polynomial place [26]. We will informally refer to the proposition that the universe is a polynomial place in the computational sense as well as the communication sense by the expression “the world is not hard enough” (WNHE) [21]. This is closely related to what is sometimes called the strong Church-Turing thesis, which asserts that any “reasonable” model of computation can be efficiently simulated on a probabilistic Turing machine [27].

Given a black box binary function $f(j)$ over n bits, the problem (called Boolean Satisfiability or SAT) of determining if there is an input j for which $f(j) = 0$, is **NP**-complete. However, if FW can collapse an outcome deterministically, one can obtain an algorithm that efficiently solves **NP**-complete problems. One prepares in polynomial time a quantum state $|\Psi'\rangle$ as follows:

$$\frac{1}{2^{n/2}} \sum_j |j\rangle|0\rangle \xrightarrow{H} |\Psi\rangle \equiv \frac{1}{2^{n/2}} \sum_j |j\rangle|f(j)\rangle, \quad (2)$$

where the second register is a qubit. One then uses FW to direct the second register to collapse to $|1\rangle$. By virtue of weak compatibility, the above decision problem about $f(j)$ is answered in the affirmative if and only if the second register returns $|1\rangle$.

We conclude that WNHE is valid in the objective world, when subjective or sub-physical effects are excluded. Even otherwise, it is a very good approximation, since the quantum coherence required to manifest nonlinear violations of WNHE, are confined to sub-neuronal scales. Further, within the objective scope, WNHE has good explanatory power [22]. The very small scale of FW-induced contravention of WNHE also present experimental challenges for its observation, an issue taken up in Section V.

V. NEUROBIOLOGICAL IMPLICATIONS AND EXPERIMENTAL TESTS

Clearly, FW-control of collapse will be at least as difficult to observe directly as wave function collapse itself. Even when brain processes are directly examined, FW-induced deviations from the Born rule will hardly be discernible from measurement errors, noise and statistical fluctuations. Any trace of FW will obviously be absent in the macroscopic ‘classical’ world, where, for a reason that famously continues to be debated (but we won’t get into here), we don’t get to see quantum superpositions [28]. The challenge to neuroscience technology will be to tease out patterns attributable to FW from the frenzied buzz of neuronal activity in the brain. This will no doubt require advanced experimental technology and sophisticated signal processing.

We term the idea that the effect of FW remains veiled behind quantum and classical noise as the *cognitive censorship*, to wit, *the mind remains hidden in matter*. Because of this, the objective universe appears to be self-contained, with no definitive indication of the subjective reality beyond, in the corpus of known experiments verifying QM. In particular, the predicted violation of energy conservation, no-signaling and WNHE at the sub-neuronal level are, per current experimental technology, are barely observable. It is an interesting question whether cognitive censorship will make FW empirically unverifiable. Here we take the optimistic position that FW is in principle not beyond the scope of experiments. In the remaining part of this Section, we point out how recent neuroscientific experiments are in agreement with the ASC model, and indicate possible ways of neurobiologically testing our proposed model, and more broadly, the idea of FW-induced wavefunction collapse.

a. Direct observation of neurons. Any spontaneous action of a human or animal subject can presumably be traced backwards along a deterministic observable causal chain, e.g., from nerve endings in the muscles controlling eye brow movement, through nerves back to a specific area in the motor cortex. In the conventional view, the brain is a complex input-output device, with voluntary actions arising from external stimuli. On the other hand, the ASC model implies that, though the brain acts as an input-output processor, yet FW makes the brain a creative device that can generate new information. The output need not be entirely explained in terms of input. Because the three stages of ASC require quantum coherence to be present during their execution, ASC implies that the above causal chain will originate from a single neuron—rather than a multi-cellular neural circuit—in the motor cortex area of the brain. We term this the ‘opening neuron’. Even if a neural circuit is recognized as closely associated with the origin of a particular voluntary action, at its core there will presumably be a single neuron that ‘wills’ the action.

No doubt, identifying such single opening neurons will be difficult, given the high density of neuronal packing, and the weakness of excitatory synapses, which would make the role of individual neurons in the mammalian cortex hard to identify. A remarkable study where the twitching of a mouse’s whisker has been traced to single pyramidal neurons in the cortex is reported in Ref. [29].

According to the ASC model, the opening neuron is the seat of FW and new physics. Experiments may be designed to verify that the cells manifest spontaneous, indeterministic behavior such as sub-neuronal variations not attributable to any external stimuli, or other, less direct signatures. The next step will be to use neurobiology and genetics to localize and understand the neuronal organelle and concomitant brain circuits responsible for the spontaneous behavior.

b. Brain scan studies through scalp electrodes, fMRI, etc. Although studies based on fMRI or scalp electrodes cannot access the quantum regime, they may be able to corroborate, modify or reject aspects of the ASC model. Based on a study on volunteers wearing scalp electrodes, Libet and collaborators [30] showed in 1983 that a ‘readiness potential’ (RP) was detected a few tenths of second before the subjects, in their own reckoning, made the decision to perform an action (to flex a finger or wrist). They interpreted their result as indicating that the motor cortex was preparing for the action, that unconscious neural processes determine actions and hence that FW was illusory.

In a recent comment on the experiment, Miller and Trevena [31] asked subjects to wait for an audio tone before making a decision to tap a key or not. If the activity detected by Libet really was the making of the decision prior to any conscious awareness of doing so, then that activity ought to occur only if the subjects decided to act. But

no such correlation was found. Miller and Trevena conclude that the RP may only indicate that the brain is paying attention and does not indicate that a decision has been made.

In the light of the ASC model, we can interpret such experiments as follows. The RP corresponds to stage 1 (at time t_1), where the brain pays attention, creating a superposition of the form:

$$|\Phi\rangle \equiv \alpha|\text{tap key}\rangle + \sqrt{1-\alpha^2}|\text{don't tap key}\rangle, \quad 0 \leq |\alpha| \leq 1. \quad (3)$$

which is presented to the mind. At time t_2 , the subject's intellect makes a decision to select one of the two alternatives (stage 2). At a slightly later time, t_3 , the mind collapses the state vector $|\Phi\rangle$ to the chosen alternative (stage 3). Our model thus explains the time gap between the RP blip (t_1) and eventual decision (t_2) in these experiments, and is compatible with the interpretation of Trevena and Miller [31] that the state at the RP stage is independent of the eventual decision made.

c. Macroscopic considerations Although the deviations from the Born rule are microscopic effects, they are cascaded into macroscopic actions of human observers, becoming amplified by nonlinear chaotic processes in the brain. This is similar to the situation that though the collapse of photon to one of two paths in a beam splitter is not directly observable, its effect amplified to classical levels is visible as the detection by a simple (avalanche) detector. Thus, the ASC model entails that the effect of FW in sentient beings will be an irreducible residual behavior that cannot be accounted for by external stimuli. There will be variability in this residual behavior, implying that the brain cannot be modeled as a complex input-output device, but as a source of information creation. Unfortunately, demonstrating such variable, spontaneous behavior experimentally can be difficult, since animals' responding differently even to the same external stimuli would normally be attributed to random errors in the complex brains.

The ASC model implies, however, that because animals should in fact possess first-order FW (however rudimentary), subjective variables responsible for free-willed behavior will cause the above variability to depart from a random noiselike pattern.

In remarkable research reported in Ref. [32], fruit flies (*Drosophila melanogaster*) were tethered in completely uniform white surroundings and their turning behavior recorded. In this setup, the flies do not receive any visual cues from the environment and since they are fixed in space, their turning attempts have no effect. Thus lacking any input, their behavior should resemble random noise, similar to a radio detuned from any station. However, the analysis showed that the temporal structure of fly behavior follows a Lévy-like probabilistic behavior pattern rather than resemble random noise, in conformance with what the ASC model suggests.

Further tests would be to subject such flies to uniform external stimuli, such as soft music, and monitor the variability. We expect that such mild stimuli will not directly affect the brain region responsible for the above motions, and thus not affect variability pattern if its origin were random errors in the brain. On the other hand, the music will presumably affect the subjective state of the flies. Therefore, if the pattern has its origin in subjective variables, we can predict a corresponding change in the variability's pattern.

d. Computational complexity approach Another approach, based on the results of Section IV B, is to study whether there are subconscious activities in the brain that correspond to problems that are suspected to be intractable from the (quantum) computation theoretic perspective, and yet the brain demonstrably solves them efficiently. As the ability to control wave-function collapse would bestow such super-Turing power (as noted in Section IV), the discovery of such cognitive processes would support the ASC model.

For example, there are several intractable problems in pattern recognition [33]. Humans, and perhaps even animals, are known to recognize patterns, such as faces, in new and unfamiliar situations quickly and far more reliably than the best computer algorithms to date. If these real life instances of subconscious problem solving can be reduced to NP-hard, and it can be experimentally verified that it is done efficiently by human subjects, this would serve as evidence of the ASC model.

VI. WHENCE FREE WILL?

Why does FW exist? A conservative view would be that it enables an organism to deviate from the local gradient, and to better than locally optimize in the struggle for self-preservation, which would require improvisation under novel situations; once this was done, FW brought along with it potential by-products, like altruism, self-harm, etc., behaviors which are not necessarily conducive to self-preservation, and hence not 'intended' by the evolutionary process that gave rise to FW. Another possibility is that FW is an emergent phenomenon that is a product of self-organization in complex quantum physical systems.

Or, one can be more adventurous and regard FW as a primitive principle in Nature, apart from the randomness and determinism. We attribute the probabilistic character of QM to it, since a deterministic dynamics could not

be weakly compatible with FW. This approach provides John Wheeler's Really Big Question "Why the quantum?" with a teleological answer: to accommodate FW. Perhaps QM is the simplest indeterministic dynamics that can accommodate FW. To be precise, this is applicable only to interpretations of QM that involve wave function collapse. They are most convenient as FW can provide a mechanism for collapse, and be weakly compatible with fundamental indeterminism that collapse models postulate. Interesting models of collapse have been proposed by various authors. In Ref. [34], we suggest that collapse is not a dynamical but an information theoretic process, precipitated by the build-up of sufficiently large entanglement, which results in amplitudes becoming unresolvable within the finite (though in principle unbounded) computational and memory resources available in Nature.

By contrast, in non-collapse interpretations of QM, such as the Many-Worlds Interpretation or the Bohmian Interpretation [35], there is no room for (non-illusory) FW. At least the Bohmian interpretation leaves open the possibility that FW could interface with the deterministic equations by influencing the initial probability distribution, but there seems to be no such redeeming feature for the Many-worlds interpretation [36]. It is worth stressing that the subjective degrees of freedom that determine the outcome of wavefunction collapse should not be confused with hidden variables in the sense of Bohm. The latter represents a device to turn QM into a deterministic theory, whereas in our approach requires, FW requires indeterminism at the objective level. To the question raised in the title of this Section, then, we are able to respond: FW comes from the same heaven that bestowed us with quantum randomness.

To be aware of a theory is to be able to talk about it and to understand its implications and limitations. To this end, one requires a meta-theory, which is equipped with a language in which to formalize propositions about the theory. An axiomatic system that is rich enough to encompass the meta-theory will, in general, be more powerful than the one that axiomatizes the theory. An instance of this is provided by Gödel's theorem [12], which provides a metamathematical proof of mathematically undecidable propositions.

If FW did not exist, and the behavior of observers were entirely determined by the rules of the base theory, then the observers' algorithmic complexity [38] would not be greater than that of the theory, making them incapable of encompassing the meta-theory. Equipped with FW, human experimenters can be *meta-entities* in the theory. It allows them to freely pose questions, perform tests and draw inferences about the theory as external agents whose choices are not entirely determined by the theory. From this perspective, FW, may be necessary for observers to be aware of a physical theory such as QM. Future work should make these speculative reflections more rigorous, and provide concrete instances of cognitive phenomena being seen in this light. A somewhat related example is Ref. [37], where we provide a concrete instance where Gödel's theorem may have relevance for quantum gravity.

VII. CONCLUSIONS

A novel insight that emerges here is that FW and, more generally, consciousness, may now be amenable to quantitative analysis in a way that is in tune with the intuitive and philosophical approach to these issues. If we regard FW as a fundamental primitive and as an axiom in a formalization of quantum theory, then quantum indeterminism finds a natural explanation. It is worth stressing that though FW provides a mechanism for wave function collapse here, it is not proposed as a means to resolve the quantum measurement problem. Our preferred approach to this problem was discussed in Section VI.

The possibility of FW-based intervention, which depends on subjective degrees of freedom, implies that QM is not universal. The physical actions of free-willed agents do not have an entirely physical explanation: the world is not closed under quantum physics. Our work shows that the study of the origin and existence of FW has ramifications for fields as diverse as neuroscience, computer science, physics, philosophy, mathematical logic and statistics. It potentially opens up many new directions of research.

The two main immediate directions suggested by our work are, on the experimental side, the neurobiological tests discussed in Section V, and on the theoretical side, development of a mathematical model for the mechanism by which subjective degrees of freedom couple to, and are able to collapse, the wavefunction of objective degrees of freedom. We revisit this issue in a following work [10].

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